

THE APPLICATIONS OF THE GAUGE TRANSFORMATION FOR THE BKP HIERARCHY

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ABSTRACT. In this paper, we investigated four applications of the gauge transformation for the BKP hierarchy. Firstly, it is found that the orbit of the gauge transformation for the constrained BKP hierarchy defines a special $(2+1)$ -dimensional Toda lattice equation structure. Then the tau function of the BKP hierarchy generated by the gauge transformation is showed to be the Pfaffian. And the higher Fay-like identities for the BKP hierarchy is also obtained through the gauge transformation. At last, the compatibility between the additional symmetry and the gauge transformation of the BKP hierarchy is proved.

Keywords: gauge transformation, the BKP hierarchy, Pfaffian, Fay-like identity, additional symmetry

PACS: 02.30.Ik

2010 MSC: 35Q53, 37K10, 37K40

1. INTRODUCTION

The gauge transformation [1, 2] provides a simple way to construct solutions for integrable hierarchies. By now, the gauge transformations of many integrable hierarchies have been constructed, for example, the KP hierarchy [1, 2], the constrained KP hierarchy [2–6], the constrained BKP and CKP hierarchy [7–10] (cBKP and cCKP), the discrete KP hierarchy [11, 12], the q -KP hierarchy [13, 14] and so on. In this paper, we will mainly study the gauge transformation for the BKP hierarchy.

The BKP hierarchy [15] is one of the most important sub-hierarchies of the KP hierarchy defined by restricting the Lax operator ${}^1L^* = -\partial L \partial^{-1}$. In order to keep the restrictions on the Lax operator of the BKP hierarchy, one cannot do gauge transformation for the BKP hierarchy by only using one kind of the basic gauge transformation operators: the differential type T_D and the integral type T_I developed by Chau *et al* in [1], and instead has to use the combination of T_D and T_I [7–10]. In particular, the generating functions of the combination of T_D and T_I are dependent on each other, which is also requested by the restriction of the Lax operator. As for the cBKP hierarchy, besides the usual BKP constraint on the Lax operator, another constraint defined with the eigenfunctions (see (19)) is also needed. So to ensure this additional constraint for the cBKP hierarchy, the corresponding

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¹Here $\partial = \partial_x$ and the asterisk stands for the conjugation operation: $(AB)^* = B^*A^*$, $\partial^* = -\partial$, $f^* = f$ with f be a function.

gauge transformation operator can be the one of the usual BKP hierarchy just by letting the generating function be one of the original eigenfunctions in the definition.

Furthermore, it is an interesting problem to explore the non-trivial relevance of the gauge transformation to the other integrable properties of the BKP besides the explicit solutions, which shall be illustrated from the following four concerns. We firstly show that the orbit of the gauge transformation for the constrained BKP hierarchy defines a special $(2 + 1)$ -dimensional Toda lattice equation structure (see (46)), proposed by Cao *et al* in [16]. This equation has some importance in mathematics and physics, whose continuous analogue is equivalent to the Ito equation [17]. And some interesting integrable properties of this special $(2 + 1)$ -dimensional Toda lattice equation can be seen in [16–19].

Then starting from the Grammian determinant solutions of the BKP hierarchy, we derived two types of the Pfaffian structures [20, 21] for the BKP hierarchy (see Proposition 4). Since the BKP hierarchy is the subhierarchy of the KP hierarchy, there are two types of the tau functions for the BKP hierarchy [15]: one is inherited from the KP hierarchy (denoted by τ_{KP}) by cancelling the even flow variables, the other is of its own (denoted by τ) defined directly by the odd variables according to the flow equations. In [9], the transformed tau function under the gauge transformation of the BKP hierarchy is only provided for τ_{KP} in the form of the Grammian determinant [22], while the transformed tau function of its own is not considered. The crucial point to derive the transformed tau function of BKP's own is to root the Grammian determinant according to the relation of these two types of the tau functions [15]. In this paper, we derived the two types of the Pfaffian structures for the BKP hierarchy, according to the even or odd steps of the gauge transformation. In particular, for the case of the even times, it can be obtained directly by using the definition of the Pfaffian, while the case of the odd times is some complicated.

Further through the gauge transformation of the BKP hierarchy, the higher order Fay-like identities [23–25] are derived. The transformed tau function under the $k + 1$ -step gauge transformation can be expressed in two different ways. One is starting from the spectral representation for the eigenfunction of the BKP hierarchy [26], then after the application of the k -step gauge transformation, the recurrence relation for the transformed tau function can be obtained. The first expression is derived by solving this recurrence relation. The other is got by expressing the eigenfunctions in the Pfaffian structure for the transformed tau function through the spectral representation [26]. The comparison of these two different expressions gives rise to the higher Fay-like identities.

At last, the compatibility between the additional symmetry and the gauge transformation of the BKP hierarchy is studied. The additional symmetry [25, 27–34] is a kind of symmetry depending explicitly on the space and time variables, involved in so-called string equation and the generalized Virasoro constraints in matrix models of the 2d quantum gravity (see [23] and references therein). In order to show a new inner consistency of the BKP integrable hierarchy, it is an interesting problem to investigate the compatibility between the gauge transformation and the additional symmetry.

This paper is organized in the following way. In section 2, some backgrounds about the BKP hierarchy are presented. Then, the gauge transformations of the BKP and cBKP hierarchies are reviewed and the orbits of the gauge transformation for BKP hierarchy are studied in Section 3. Then, the Pfaffian structures of the BKP hierarchy are investigated in Section 4. Further, the higher order Fay-like identities through the gauge transformation is derived in Section 5. And in Section 6, the compatibility of the additional symmetry and the gauge transformation for the BKP hierarchy is checked. At last, we devote section 7 to some conclusions and discussions.

2. BACKGROUNDS ON THE BKP HIERARCHY

In this section, we shall review some backgrounds of the BKP hierarchy [15]. The BKP hierarchy is defined in Lax form as follows

$$\partial_{2n+1}L = [B_{2n+1}, L], \quad B_{2n+1} = (L^{2n+1})_+, \quad n = 0, 1, 2, \dots, \quad (1)$$

where the Lax operator is given by

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \dots, \quad (2)$$

with the coefficient functions u_i depending on the time variables $t = (t_1 = x, t_3, t_5, \dots)$, and satisfies the BKP constraint

$$L^* = -\partial L \partial^{-1}, \quad (3)$$

which is equivalent to the condition

$$B_{2n+1}(1) = 0. \quad (4)$$

Here $\partial_{2n+1} = \partial_{t_{2n+1}}$. And for any (pseudo-) differential operator A , B , and a function f , $(A)_\pm$ denote the differential part and the integral part of the pseudo-differential operator A respectively. The symbol $A(f)$ will indicate the action of A on f , whereas the symbol Af will denote just operator product of A and f . The Lax equation (1) is equivalent to the compatibility condition of the linear system

$$L(\psi_{BA}(t, \lambda)) = \lambda \psi_{BA}(t, \lambda), \quad \partial_{2n+1} \psi_{BA}(t, \lambda) = B_{2n+1}(\psi_{BA}(t, \lambda)), \quad (5)$$

where $\psi_{BA}(t, \lambda)$ is called the Baker-Akhiezer (BA) wave function.

The whole hierarchy can be expressed in terms of a dressing operator W , so that

$$L = W \partial W^{-1}, \quad W = 1 + \sum_{j=1}^{\infty} w_j \partial^{-j}, \quad (6)$$

and the Lax equation is equivalent to the Sato's equation

$$\partial_{2n+1} W = -(L^{2n+1})_- W, \quad (7)$$

with constraint

$$W^* \partial W = \partial. \quad (8)$$

Let the solutions of the linear system (5) be the form

$$\psi_{BA}(t, \lambda) = W(e^{\xi(t, \lambda)}) = w(t, \lambda)e^{\xi(t, \lambda)} \quad (9)$$

where $\xi(t, \lambda) = \sum_{i=0}^{\infty} t_{2i+1} \lambda^{2i+1}$ and $w(t, \lambda) = 1 + w_1/\lambda + w_2/\lambda^2 + \dots$. Then $\psi_{BA}(t, \lambda)$ is a wave function of the BKP hierarchy if and only if it satisfies the bilinear identity [15]

$$\int d\lambda \lambda^{-1} \psi_{BA}(t, \lambda) \psi_{BA}(t', -\lambda) = 1, \quad \forall t, t', \quad (10)$$

where $\int d\lambda \equiv \oint_{\infty} \frac{d\lambda}{2\pi i} = \text{Res}_{\lambda=\infty}$ and $t = (t_1 = x, t_3, t_5, \dots)$.

From the bilinear identity (10), solutions of the BKP hierarchy can be characterized by a single function $\tau(t)$ called τ -function such that [15]

$$\psi_{BA}(t, \lambda) = \frac{\tau(t - 2[\lambda^{-1}])}{\tau(t)} e^{\xi(t, \lambda)}, \quad (11)$$

where $[\lambda^{-1}] = (\lambda^{-1}, \frac{1}{3}\lambda^{-3}, \dots)$. This implies that all dynamical variables $\{u_i\}$ in the Lax operator L can be expressed by τ -function. Moreover, another important property of τ function of the BKP is the following Fay like identity [25]

$$\begin{aligned} & \sum_{(s_1, s_2, s_3)} \frac{(s_1 - s_0)(s_1 + s_2)(s_1 + s_3)}{(s_1 + s_0)(s_1 - s_2)(s_1 - s_3)} \tau(t + 2[s_2] + 2[s_3]) \tau(t + 2[s_0] + 2[s_1]) \\ & + \frac{(s_0 - s_1)(s_0 - s_2)(s_0 - s_3)}{(s_0 + s_1)(s_0 + s_2)(s_0 + s_3)} \tau(t + 2[s_0] + 2[s_1] + 2[s_2] + 2[s_3]) \tau(t) = 0, \end{aligned} \quad (12)$$

where (s_1, s_2, s_3) stands for cyclic permutations of s_1, s_2 and s_3 , and the differential Fay identity [25]

$$\begin{aligned} & \left(\frac{1}{s_2^2} - \frac{1}{s_1^2} \right) \{ \tau(t + 2[s_1]) \tau(t + 2[s_2]) - \tau(t + 2[s_1] + 2[s_2]) \tau(t) \} \\ & = \left(\frac{1}{s_2} + \frac{1}{s_1} \right) \{ \partial \tau(t + 2[s_2]) \tau(t + 2[s_1]) - \partial \tau(t + 2[s_1]) \tau(t + 2[s_2]) \} \\ & \quad + \left(\frac{1}{s_2} - \frac{1}{s_1} \right) \{ \tau(t + 2[s_1] + 2[s_2]) \partial \tau(t) - \partial \tau(t + 2[s_1] + 2[s_2]) \tau(t) \}. \end{aligned} \quad (13)$$

In the BKP hierarchy, if Φ (or Ψ) satisfies

$$\partial_{2n+1} \Phi = B_{2n+1}(\Phi) \quad (\text{or } \partial_{2n+1} \Psi = -B_{2n+1}^*(\Psi)), \quad n = 0, 1, 2, \dots, \quad (14)$$

we shall call Φ (or Ψ) eigenfunction (or adjoint eigenfunction) of the BKP hierarchy. Obviously, by (4) and (5), 1 and $\psi_{BA}(t, \lambda)$ are the eigenfunctions. From the fact $B_{2n+1}^* \partial = -\partial B_{2n+1}$, any adjoint eigenfunction Ψ can be given in the form of $\Psi = \Phi_x$ with Φ be an eigenfunction. In particular, the adjoint BA function $\psi_{BA}^*(t, \lambda) = -\lambda^{-1} \psi_{BA}(t, -\lambda)_x$, where

$$\psi_{BA}^*(t, \lambda) \equiv W^{*-1}(e^{-\xi(t, \lambda)}). \quad (15)$$

Thus in the BKP case, it is enough to only consider the eigenfunctions. The relation between the eigenfunction Φ and the BA wave function $\psi_{BA}(t, \lambda)$ is showed in the following spectral representation [26],

$$\Phi(t) = \int d\lambda \psi_{BA}(t, \lambda) \varphi(\lambda), \quad (16)$$

with $\varphi(\lambda) = \lambda^{-1} S(\Phi(t'), \psi_{BA}(t', -\lambda)_{x'})$, where for any pair of (adjoint) eigenfunctions $\Phi(t), \Psi(t)$, $S(\Phi(t), \Psi(t))$ is determined by the following equations,

$$\frac{\partial}{\partial t_{2n+1}} S(\Phi(t), \Psi(t)) = \text{Res}(\partial^{-1} \Psi B_{2n+1} \Phi \partial^{-1}), \quad n = 0, 1, 2, 3, \dots \quad (17)$$

In particular for $n = 0$,

$$\partial_x S(\Phi(t), \Psi(t)) = \Phi(t) \Psi(t). \quad (18)$$

The constrained BKP hierarchy [21, 35] can be defined by restricting the Lax operator of the BKP hierarchy (1) in the following form:

$$L^{2k+1} = \partial^{2k+1} + \sum_{i=1}^{2k-1} v_i \partial^i + \sum_{j=1}^m (\Phi_{1j} \partial^{-1} \Phi_{2j,x} - \Phi_{2j} \partial^{-1} \Phi_{1j,x}), \quad (19)$$

for some fixed k , where Φ_{1j} and Φ_{2j} are the eigenfunctions satisfying (14).

3. THE GAUGE TRANSFORMATION FOR THE BKP HIERARCHY

Now let's review some results about the gauge transformation for the BKP hierarchy [7–10]. Assume T be a pseduo-differential operator and $L^{(1)} = T L^{(0)} T^{-1}$ with $L^{(0)}$ be the Lax operator of the BKP hierarchy. T is called the gauge transformation for the BKP hierarchy, if it satisfies:

- preserving the Lax equation: $\partial_{2n+1} L^{(1)} = [B_{2n+1}^{(1)}, L^{(1)}]$, $B_{2n+1}^{(1)} = (L^{(1)})_+^{2n+1}$,
- preserving the BKP constraint: $L^{(1)*} = -\partial L^{(1)} \partial^{-1}$.

In [7–9], it is showed that the operator $T(\Phi) = T_I(\Phi/2) T_D(\Phi)$ satisfies the two conditions above, and thus is the gauge transformation operator of the BKP hierarchy, where Φ is the eigenfunctions of the BKP hierarchy and

$$T_D(\Phi) = \Phi \partial \Phi^{-1}, \quad T_I(\Phi/2) = (\Phi/2)^{-1} \partial^{-1} (\Phi/2). \quad (20)$$

There is an important property satisfied by $T(\Phi) = T_I(\Phi/2) T_D(\Phi)$ listed in the lemma below [8].

Lemma 1. *The operator $T(\Phi) = T_I(\Phi/2) T_D(\Phi)$ obeys*

$$T(\Phi)^* \partial T(\Phi) = \partial. \quad (21)$$

Under the transformation $T(\Phi) = T_I(\Phi/2) T_D(\Phi)$, we will have

$$L^{(0)} \rightarrow L^{(1)} = T(\Phi) L^{(0)} T(\Phi)^{-1}, \quad (22)$$

$$\chi^{(0)} \rightarrow \chi^{(1)} = T(\Phi) (\chi^{(0)}), \quad (23)$$

$$\tau_{KP}^{(0)} \rightarrow \tau_{KP}^{(1)} = \frac{\Phi^2}{2} \tau_{KP}^{(0)}, \quad (24)$$

where $L^{(0)}$ is the initial Lax operator of the BKP hierarchy, $\chi^{(0)}$ is the initial eigenfunction, and $\tau_{KP}^{(0)}$ is the initial BKP tau function inherited from the KP hierarchy, which satisfies [15]

$$\tau_{KP} = \tau^2. \quad (25)$$

From (24) and (25),

$$\tau^{(0)} \rightarrow \tau^{(1)} = \frac{\Phi}{\sqrt{2}} \tau^{(0)}. \quad (26)$$

The actions of the gauge transformation $T(\Phi) = T_I(\Phi/2)T_D(\Phi)$ on the dressing operator and the BA function are showed as follows.

$$W^{(0)} \rightarrow W^{(1)} = T(\Phi)W^{(0)}, \quad (27)$$

$$\psi_{BA}^{(0)} \rightarrow \psi_{BA}^{(1)} = T(\Phi)(\psi_{BA}^{(0)}), \quad (28)$$

Therefore under the successive gauge transformation

$$T_n = T(\Phi_n^{(n-1)})T(\Phi_{n-1}^{(n-2)})...T(\Phi_2^{(1)})T(\Phi_1^{(0)}), \quad (29)$$

where $\{\Phi_1, \Phi_2, ..., \Phi_n\}$ are a group of different eigenfunctions of the BKP hierarchy, the Lax operator $L^{(0)}$ and the BKP tau function $\tau^{(0)}$ will transform as

$$L^{(n)} = T_n L^{(0)} T_n^{-1}, \quad (30)$$

$$\tau_{KP}^{(n)} = G_n(\Phi_1, \Phi_2, ..., \Phi_n) \tau_{KP}^{(0)}, \quad (31)$$

$$\tau^{(n)} = \left(\frac{1}{2}\right)^{\frac{n}{2}} \Phi_n^{(n-1)} \Phi_{n-1}^{(n-2)} ... \Phi_2^{(1)} \Phi_1^{(0)} \tau^{(0)}, \quad (32)$$

where

$$G_n(\Phi_1, \Phi_2, ..., \Phi_n) = \det(S(\Phi_i, \Phi_{jx}))_{1 \leq i, j \leq n}. \quad (33)$$

The gauge transformation for the cBKP hierarchy is constructed in [9] from the corresponding results about the cKP hierarchy by considering the BKP constraint. Now let's review the results about the gauge transformation for the cBKP hierarchy.

Assume

$$\left(L^{(0)}\right)^{2k+1} = \left(L^{(0)}\right)_+^{2k+1} + \sum_{i=1}^m \left(\Phi_{1i}^{(0)} \partial^{-1} \left(\Phi_{2i}^{(0)}\right)_x - \Phi_{2i}^{(0)} \partial^{-1} \left(\Phi_{1i}^{(0)}\right)_x\right) \quad (34)$$

for the initial Lax operator of the cBKP hierarchy. Under the BKP gauge transformation operator $T(\chi) = T_I(\chi)T_D(\chi)$, (34) will become into:

$$\begin{aligned} \left(L^{(1)}\right)^{2k+1} &= T(\chi) \left(L^{(0)}\right)^{2k+1} T_I(\chi)^{-1} = \left(L^{(1)}\right)_+^{2k+1} + \left(L^{(1)}\right)_-^{2k+1}, \\ \left(L^{(1)}\right)_-^{2k+1} &= \Phi_{1,0}^{(1)} \partial^{-1} \left(\Phi_{2,0}^{(1)}\right)_x - \Phi_{2,0}^{(1)} \partial^{-1} \left(\Phi_{1,0}^{(1)}\right)_x \end{aligned} \quad (35)$$

$$+ \sum_{i=1}^m \left(\Phi_{1i}^{(1)} \partial^{-1} \left(\Phi_{2i}^{(1)} \right)_x - \Phi_{2i}^{(1)} \partial^{-1} \left(\Phi_{1i}^{(1)} \right)_x \right), \quad (36)$$

$$\Phi_{1,0}^{(1)} = \left(T(\chi) \left(L^{(0)} \right)^{2k+1} \right) \left(\Phi_{1,0}^{(0)} \right), \quad \Phi_{2,0}^{(1)} = -\frac{2}{\Phi_{1,0}^{(0)}}, \quad (37)$$

$$\Phi_{1i}^{(1)} = T(\chi) \left(\Phi_{1i}^{(0)} \right), \quad \Phi_{2i}^{(1)} = T(\chi)^{* -1} \left(\Phi_{2i}^{(0)} \right), \quad (38)$$

In order to preserve the form of (34), χ is required to coincide with one of the original eigenfunctions in $(L^{(0)})^{2k+1}$, e.g. $\chi = \Phi_{1,1}^{(0)}$, since $\Phi_{1,1}^{(1)} = 0$ in this case. Applying successive the BKP gauge transformations

$$L^{(n+1)} = T^{(n)} L^{(n)} \left(T^{(n)} \right)^{-1}, \quad T^{(n)} \equiv T(\Phi_{1,1}^{(n)}) \quad (39)$$

yields:

$$\left(L^{(n+1)} \right)^{2k+1} = \left(L^{(n+1)} \right)_+^{2k+1} + \left(L^{(n+1)} \right)_-^{2k+1}, \quad (40)$$

$$\left(L^{(n+1)} \right)_-^{2k+1} = \sum_{i=1}^m \left(\Phi_{1i}^{(n+1)} \partial^{-1} \left(\Phi_{2i}^{(n+1)} \right)_x - \Phi_{2i}^{(n+1)} \partial^{-1} \left(\Phi_{1i}^{(n+1)} \right)_x \right), \quad (41)$$

$$\Phi_{1,1}^{(n+1)} = \left(T^{(n)} \left(L^{(n)} \right)^{2k+1} \right) \left(\Phi_{1,1}^{(n)} \right), \quad \Phi_{2,1}^{(n+1)} = -\frac{2}{\Phi_{1,1}^{(n)}}, \quad (42)$$

$$\Phi_{1i}^{(n+1)} = T^{(n)} \left(\Phi_{1i}^{(n)} \right), \quad \Phi_{2i}^{(n+1)} = \left(T^{(n)} \right)^{* -1} \left(\Phi_{2i}^{(n)} \right), \quad i = 2, 3, 4, \dots \quad (43)$$

$$\tau^{(n+1)} = \frac{\Phi_{1,1}^{(n)}}{\sqrt{2}} \tau^{(n)}. \quad (44)$$

We next restrict the cBKP hierarchy (19) to $m = 1$ case and then will find the following proposition:

Proposition 2. *The orbit of the gauge transformation for the cBKP hierarchy (19) (for $k \geq 1$ and $m = 1$) defines a special $(2+1)$ -dimensional Toda lattice equation structure (here we set $\Phi = \Phi_{1,1}$).*

$$\partial_x \partial_{2k+1} \ln \Phi^{(n)} = \frac{(\Phi^{(n)} \Phi^{(n+1)})_x}{(\Phi^{(n)})^2} + \left(\left(\Phi^{(n-1)} \right)^{-1} \right)_x \Phi^{(n)} - \left(\Phi^{(n)} \right)_x \left(\Phi^{(n-1)} \right)^{-1}, \quad n = 0, 1, 2, \dots \quad (45)$$

Proof. Firstly from (41) and (42),

$$\begin{aligned} \Phi_{1,1}^{(n+1)} &= T^{(n)} \left(\left(L^{(n)} \right)_+^{2k+1} \left(\Phi_{1,1}^{(n)} \right) \right) + T^{(n)} \left(\left(L^{(n)} \right)_-^{2k+1} \left(\Phi_{1,1}^{(n)} \right) \right) \\ &= T^{(n)} \left(\partial_{2k+1} \left(\Phi_{1,1}^{(n)} \right) \right) + T^{(n)} \left(\Phi_{1,1}^{(n)} \int^x \left(\Phi_{2,1}^{(n)} \right)_x \Phi_{1,1}^{(n)} - \Phi_{2,1}^{(n)} \int^x \left(\Phi_{1,1}^{(n)} \right)_x \Phi_{1,1}^{(n)} \right) \end{aligned}$$

Then multiplying $\Phi_{1,1}^{(n)}$ to the both sides of the above identity and differentiating it with respect to x , by the definition of $T^{(n)}$, we have

$$\frac{(\Phi_{1,1}^{(n)} \Phi_{1,1}^{(n+1)})_x}{(\Phi_{1,1}^{(n)})^2} = \partial_x \partial_{2k+1} \ln \left(\Phi_{1,1}^{(n)} \right) + \left(\Phi_{2,1}^{(n)} \right)_x \Phi_{1,1}^{(n)} - \frac{1}{2} \partial_x \left(\Phi_{2,1}^{(n)} \Phi_{1,1}^{(n)} \right)$$

$$\begin{aligned}
&= \partial_x \partial_{2k+1} \ln \left(\Phi_{1,1}^{(n)} \right) + \frac{1}{2} \left(\Phi_{2,1}^{(n)} \right)_x \Phi_{1,1}^{(n)} - \frac{1}{2} \left(\Phi_{1,1}^{(n)} \right)_x \Phi_{2,1}^{(n)} \\
&= \partial_x \partial_{2k+1} \ln \left(\Phi_{1,1}^{(n)} \right) - \left(\left(\Phi_{1,1}^{(n-1)} \right)^{-1} \right)_x \Phi_{1,1}^{(n)} + \left(\Phi_{1,1}^{(n)} \right)_x \left(\Phi_{1,1}^{(n-1)} \right)^{-1}
\end{aligned}$$

At last by considering $\Phi = \Phi_{1,1}$, (45) will be derived. \square

Note that (45) if set $\Phi^{(n)} = e^{u(n)}$ and $t_{2k+1} = y$, then we will get the following special $(2+1)$ -dimensional Toda lattice equation [16–19]

$$\partial_x \partial_y u(n) = e^{u(n+1)-u(n)} \partial_x (u(n+1) + u(n)) - e^{u(n)-u(n-1)} \partial_x (u(n) + u(n-1)). \quad (46)$$

4. THE PFAFFIAN STRUCTURE OF THE BKP HIERARCHY

In this section, we will investigate the Pfaffian structures of the BKP hierarchy generated by the gauge transformation.

Firstly, let's denote [21, 26]

$$\Omega(\Phi_1, \Phi_2) = S(\Phi_2, \Phi_{1x}) - S(\Phi_1, \Phi_{2x}) \quad (47)$$

for the eigenfunctions Φ_1 and Φ_2 of the BKP hierarchy, which satisfies

$$\Omega(\Phi_1, \Phi_2) = -\Omega(\Phi_2, \Phi_1). \quad (48)$$

In particular, $\Omega(\Phi_1, 1) = \Phi_1$, since 1 is also an eigenfunction of the BKP hierarchy. With the help of the spectral representation (16) and (11), we have [26]

$$\Omega(\Phi_1, \Phi_2) = \int \int d\lambda d\mu \varphi_1(\lambda) \varphi_2(\mu) e^{\xi(t, \lambda) + \xi(t, \mu)} \frac{\lambda - \mu}{\lambda + \mu} \frac{\tau(t - 2[\lambda^{-1}] - 2[\mu^{-1}])}{\tau(t)}. \quad (49)$$

Thus from (18) and (47),

$$S(\Phi_1, \Phi_{2x}) = \frac{1}{2} (\Phi_1 \Phi_2 - \Omega(\Phi_1, \Phi_2)). \quad (50)$$

Then we start from the $\tau_{KP}^{(n)}$ generated by the gauge transformation T_n (see (29)) for the BKP hierarchy. According to (48) and (50), we find

$$\frac{\tau_{KP}^{(n)}}{\tau_{KP}^{(0)}} = \frac{1}{2^n} \det(\Phi_i \Phi_j - \Omega(\Phi_i, \Phi_j))_{1 \leq i, j \leq n} = \frac{1}{2^n} \det(\Phi_i \Phi_j + \Omega(\Phi_i, \Phi_j))_{1 \leq i, j \leq n}, \quad (51)$$

where $\det(A) = \det(A^T)$ (T denote the transpose) for some matrix A has been used. Before the further discussion, the lemma [36] below is needed.

Lemma 3. *If set $A = (a_{i,j})_{1 \leq i, j, n}$, A^* be the adjoint matrix of A , and*

$$X = (x_1, x_2, \dots, x_n)^T, \quad Y = (y_1, y_2, \dots, y_n)^T,$$

then

$$\det(A + XY^T) = \det(A) + Y^T A^* X. \quad (52)$$

Thus if denote $X = (\Phi_1, \Phi_2, \dots, \Phi_n)^T$ and $\Omega = (\Omega(\Phi_i, \Phi_j))_{1 \leq i, j \leq n} \triangleq (\Omega_{ij})_{1 \leq i, j \leq n}$, then from (51) and (52), we have

$$\frac{2^n \tau_{KP}^{(n)}}{\tau_{KP}^{(0)}} = \det(\Omega) + X^T \Omega^* X = \begin{cases} \det(\Omega), & n \text{ is even;} \\ X^T \Omega^* X, & n \text{ is odd;} \end{cases} \quad (53)$$

since Ω is antisymmetric matrix, and when n is odd, Ω^* is symmetric matrix, while n is even, Ω^* is antisymmetric. Therefore according to (25), the transformed form of the BKP's own tau function τ can be derived just by taking the square root of (53).

When n is even, the transformed BKP's own tau function τ can be expressed as the Pfaffian [20, 21], which is defined as the square root of the determinat of an antisymmetric matrix of even order (see [37] for more details). For example, the Pfaffians of 2×2 and 4×4 antisymmetric matrix $A = (a_{ij})$ are $\text{Pf}(A) = a_{12}$ and $\text{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$. Thus according to (25) and (53),

$$\frac{\tau^{(n)}}{\tau^{(0)}} = \frac{(-1)^{a(n)}}{2^{n/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n), \quad (54)$$

where $\text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n) = \text{Pf}(\Omega)$. The choice of the sign of (54), when taking square root, can be seen as follows. According to (32), if letting $\Phi_{n+1} = 1$, since 1 is also the eigenfunction of the BKP hierarchy, and noting that $1^{(n)} = T_n(1) = 1^n = 1$, then

$$\frac{\tau^{(n)}}{\tau^{(0)}} = \sqrt{2} \frac{\tau^{(n+1)}}{\tau^{(0)}}, \quad (55)$$

thus $\frac{\tau^{(n)}}{\tau^{(0)}}$ and $\frac{\tau^{(n+1)}}{\tau^{(0)}}$ have the same sign. Similarly, if setting $\Phi_{n+2} = 1$, by noting that $1^{(n+1)} = -1$, then

$$\frac{\tau^{(n+1)}}{\tau^{(0)}} = -\sqrt{2} \frac{\tau^{(n+2)}}{\tau^{(0)}}, \quad (56)$$

which shows that $\frac{\tau^{(n+1)}}{\tau^{(0)}}$ and $\frac{\tau^{(n+2)}}{\tau^{(0)}}$ have the different signs. As a result, $\frac{\tau^{(n)}}{\tau^{(0)}}$ and $\frac{\tau^{(n+2)}}{\tau^{(0)}}$ have the different signs. So we can prove that

$$a(n) = \frac{n(n-1)}{2} \quad (57)$$

by induction.

When n is odd, the expression of the transformed BKP's own tau function τ is some complicated, which will be the main task of this section. For this, the first thing is to root the quadratic form of $X^T \Omega^* X$, thus we need to show that the quadratic form of $X^T \Omega^* X$ is positive semidefinite. In fact, from the fact $\det(\Omega) = 0$, we can know that $\text{rank}(\Omega) \leq n-1$, and thus $\text{rank}(\Omega^*) \leq 1$ [36]. Further according to the fact $\tau_{KP}^{(n)} \neq 0$ and (53), $\text{rank}(\Omega) = n-1$ and $\text{rank}(\Omega^*) = 1$. Thus for the $n \times n$ antisymmetric matrix Ω , there exists the orthogonal matrix Q [36], such that,

$$\Omega = Q^T B Q \quad (58)$$

where

$$B = \begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & b_s \\ & & & -b_s & 0 \\ & & & & & 0 \end{pmatrix}, \quad (59)$$

with $s = (n - 1)/2$. Therefore,

$$\Omega^* = Q^* B^* Q^{*T} = B_{nn} \alpha_n \alpha_n^T = \beta \beta^T, \quad (60)$$

with

$$B^* = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & B_{nn} \end{pmatrix}, \quad (61)$$

where $B_{nn} = b_1^2 \cdots b_s^2 \geq 0$, $Q^* = (\alpha_1, \dots, \alpha_n)$, and $\beta = b_1 \cdots b_s \alpha_n$. So the matrix Ω^* is positive semidefinite and

$$\sqrt{X^T \Omega^* X} = X^T \beta. \quad (62)$$

On the other hand, denote $D \begin{bmatrix} i \\ j \end{bmatrix}$ as the (j, k) th minor of N th-order determinant $D = \det(a_{ij})_{1 \leq i, j \leq N}$, that is, the $(N - 1)$ th-order determinant obtained by eliminating the j th row and the k th column from D , and the $(N - 2)$ nd-order determinant obtained by eliminating the j th and k th rows and the l th and m th columns from the determinant D as $D \begin{bmatrix} j & k \\ l & m \end{bmatrix}$, then the Jacobi identity [37] is expressed in the following form

$$D \begin{bmatrix} i \\ i \end{bmatrix} D \begin{bmatrix} j \\ j \end{bmatrix} - D \begin{bmatrix} i \\ j \end{bmatrix} D \begin{bmatrix} j \\ i \end{bmatrix} = D \begin{bmatrix} i & j \\ i & j \end{bmatrix} D. \quad (63)$$

Thus for the antisymmetric matrix Ω of odd order, from (63) and the fact that Ω^* is symmetric, we have

$$D \begin{bmatrix} i \\ j \end{bmatrix}^2 = D \begin{bmatrix} i \\ i \end{bmatrix} D \begin{bmatrix} j \\ j \end{bmatrix}. \quad (64)$$

Since $D \begin{bmatrix} i \\ i \end{bmatrix}$ is the determinant of the antisymmetric matrix of even order, we have

$$D \begin{bmatrix} i \\ i \end{bmatrix} = \text{Pf}^2(\Phi_1, \dots, \widehat{\Phi_i}, \dots, \Phi_n) \triangleq \text{Pf}_i^2, \quad i = 1, 2, \dots, n. \quad (65)$$

Therefore if assume $\Omega^* = (A_{ij})_{1 \leq i, j \leq n}$, then

$$A_{ij} = (-1)^{i+j} D \begin{bmatrix} j \\ i \end{bmatrix} = (-1)^{i+j} \text{Pf}_i \text{Pf}_j, \quad (66)$$

where we have used the fact Ω^* is positive semidefinite to omit the negative sign when taking the square root. So we can take

$$\beta = (-1)^{b(n)+1} (-\text{Pf}_1, \text{Pf}_2, -\text{Pf}_3, \dots, \text{Pf}_{n-1}, -\text{Pf}_n)^T. \quad (67)$$

Thus by (62),

$$\begin{aligned} \sqrt{X^T \Omega^* X} &= (-1)^{b(n)+1} \sum_{j=1}^n (-1)^j \Phi_j \text{Pf}_j = (-1)^{b(n)} \sum_{j=1}^n (-1)^{j-1} \Omega(\Phi_j, 1) \text{Pf}_j \\ &= (-1)^{b(n)} \sum_{j=1}^n (-1)^{j-1} \text{Pf}(\Phi_j, 1) \text{Pf}_j = (-1)^{b(n)} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, 1). \end{aligned} \quad (68)$$

Therefore according to (25), (53) and (68),

$$\frac{\tau^{(n)}}{\tau^{(0)}} = \frac{(-1)^{b(n)}}{2^{n/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, 1), \quad (69)$$

when n is odd. By the same way as the even case, we have

$$b(n) = \frac{n(n-1)}{2}. \quad (70)$$

At last, we summarize the result above as the following proposition.

Proposition 4. *Under the gauge transformation T_n (see (29)), the BKP tau function τ and the eigenfunction χ will transform as*

- n is even

$$\frac{\tau^{(n)}}{\tau^{(0)}} = \frac{(-1)^{\frac{n(n-1)}{2}}}{2^{n/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n), \quad (71)$$

$$\chi^{(n)} = \frac{\text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, \chi, 1)}{\text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n)}; \quad (72)$$

- n is odd

$$\frac{\tau^{(n)}}{\tau^{(0)}} = \frac{(-1)^{\frac{n(n-1)}{2}}}{2^{n/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, 1), \quad (73)$$

$$\chi^{(n)} = -\frac{\text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, \chi)}{\text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_n, 1)}. \quad (74)$$

5. THE DERIVATION OF THE FAY LIKE IDENTITIES FOR THE BKP HIERARCHY BY THE GAUGE TRANSFORMATION

In this section, we will investigate the applications of the gauge transformation in the derivation of the Fay-like identities for the BKP hierarchy.

Firstly, with the help of the spectral representation (16) for the eigenfunction, we have

$$\Phi_{k+1}^{(0)} = \int d\lambda \varphi_{k+1}^{(0)}(\lambda) \psi_{BA}^{(0)}(t, \lambda), \quad (75)$$

where $\Phi_{k+1}^{(0)}$ is the generators of the gauge transformation T_n (see (29)). Then the application of the gauge transformation T_k to the both side of (75) will lead to

$$\Phi_{k+1}^{(k)} = \int d\lambda \varphi_{k+1}^{(0)}(\lambda) \psi_{BA}^{(k)}(t, \lambda) = \int d\lambda \varphi_{k+1}^{(0)}(\lambda) \frac{\tau^{(k)}(t - 2[\lambda^{-1}])}{\tau^{(k)}(t)} e^{\xi(t, \lambda)}, \quad (76)$$

by using (11) and (28). Further according to (32), we can get the following recurrence relation,

$$\begin{aligned} \tau^{(k+1)} &= \frac{1}{\sqrt{2}} \int d\lambda \varphi_{k+1}^{(0)}(\lambda) e^{\xi(t, \lambda)} \tau^{(k)}(t - 2[\lambda^{-1}]) \\ &= \frac{1}{\sqrt{2}} \int d\lambda \varphi_{k+1}^{(0)}(\lambda) : e^{-\hat{\theta}(\lambda)} : \tau^{(k)}(t) \end{aligned} \quad (77)$$

where

$$\hat{\theta}(\lambda) = - \sum_{i=0}^{\infty} t_{2i+1} \lambda^{2i+1} + \sum_{i=0}^{\infty} \frac{2}{(2i+1)\lambda^{2i+1}} \frac{\partial}{\partial t_{2i+1}}, \quad (78)$$

and $: \dots :$ indicates standard normal ordering, i.e., $\frac{\partial}{\partial t_{2i+1}}$ is on the right of t_{2i+1} . Using the following identity:

$$: e^{-\hat{\theta}(\lambda_k)} : \dots : e^{-\hat{\theta}(\lambda_0)} := e^{-\sum_{j=0}^k \hat{\theta}(\lambda_j)} : \prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right), \quad (79)$$

we can solve the recurrence relation (77) and express $\tau^{(k+1)}$ in terms of the initial tau function $\tau^{(0)}$:

$$\begin{aligned} \tau^{(k+1)} &= \left(\frac{1}{2} \right)^{\frac{k+1}{2}} \int \prod_{j=0}^k d\lambda_j : e^{-\hat{\theta}(\lambda_k)} : \dots : e^{-\hat{\theta}(\lambda_0)} : \tau^{(0)}(t) \prod_{j=0}^k \varphi_{j+1}^{(0)}(\lambda_j) \\ &= \left(\frac{1}{2} \right)^{\frac{k+1}{2}} \int \prod_{j=0}^k d\lambda_j \prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \prod_{s=0}^k f_s(t, \lambda_s) \tau^{(0)}(t - 2 \sum_{j=0}^k [\lambda_j^{-1}]), \end{aligned} \quad (80)$$

where

$$f_k(t, \lambda) = \varphi_{k+1}^{(0)}(\lambda) e^{\xi(t, \lambda)}. \quad (81)$$

On the other hand, when k is odd, if let $k+1 = 2n$, then according to Proposition 4 and (49),

$$\begin{aligned} &\frac{\tau^{(k+1)}}{\tau^{(0)}} \\ &= \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_{k+1}) = \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \sum_P (-1)^P \Omega_{i_1, i_2} \Omega_{i_3, i_4} \dots \Omega_{i_{2n-1}, i_{2n}} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \int \prod_{j=0}^k d\lambda_j \prod_{s=0}^k f_s(t, \lambda_s) \sum_P (-1)^P \prod_{l=1}^n \left(\frac{\lambda_{i_{2l-1}} - \lambda_{i_{2l}}}{\lambda_{i_{2l-1}} + \lambda_{i_{2l}}} \right) \frac{\tau^{(0)}(t - 2[\lambda_{i_{2l-1}}^{-1}] - 2[\lambda_{i_{2l}}^{-1}])}{\tau^{(0)}(t)} \\
&= \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \int \prod_{j=0}^k d\lambda_j \prod_{s=0}^k f_s(t, \lambda_s) \text{Pf} \left(\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \frac{\tau^{(0)}(t - 2[\lambda_i^{-1}] - 2[\lambda_j^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq i, j \leq k}, \quad (82)
\end{aligned}$$

where P is the sequence selected from $\{0, 1, 2, 3, \dots, 2n-1\}$ that satisfy

$$\begin{aligned}
i_1 &< i_2, i_3 < i_4, i_5 < i_6, \dots, i_{2n-1} < i_{2n}, \\
i_1 &< i_3 < i_5 < \dots < i_{2n-1}, \quad (83)
\end{aligned}$$

and the factor $(-1)^P$ takes the value $+1$ (-1) if the sequence i_1, i_2, \dots, i_{2n} is an even (odd) permutation of $0, 1, 2, \dots, 2n-1$. Thus comparing with (80), we can get

$$\begin{aligned}
&\prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \tau^{(0)}(t - 2 \sum_{j=0}^k [\lambda_j^{-1}]) \\
&= (-1)^{\frac{k(k+1)}{2}} \tau^{(0)}(t) \text{Pf} \left(\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \frac{\tau^{(0)}(t - 2[\lambda_i^{-1}] - 2[\lambda_j^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq i, j \leq k}. \quad (84)
\end{aligned}$$

When k is even, Proposition 4, (16) and (49) will lead to

$$\begin{aligned}
&\frac{\tau^{(k+1)}}{\tau^{(0)}} \\
&= \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \text{Pf}(\Phi_1, \Phi_2, \dots, \Phi_{k+1}, 1) = \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \sum_{j=1}^{k+1} (-1)^{j-1} \Phi_j \text{Pf}_j \\
&= \frac{(-1)^{\frac{k(k+1)}{2}}}{2^{(k+1)/2}} \int \prod_{j=0}^k d\lambda_j \prod_{s=0}^k f_s(t, \lambda_s) \sum_{l=0}^k (-1)^l \frac{\tau^{(0)}(t - 2[\lambda_l^{-1}])}{\tau^{(0)}(t)} \\
&\quad \times \text{Pf} \left(\left(\frac{\lambda_p - \lambda_q}{\lambda_p + \lambda_q} \right) \frac{\tau^{(0)}(t - 2[\lambda_p^{-1}] - 2[\lambda_q^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq p, q \leq k, p \neq l, q \neq l} \quad (85)
\end{aligned}$$

The comparison with (80) gives rise to

$$\begin{aligned}
&\prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \tau^{(0)}(t - 2 \sum_{j=0}^k [\lambda_j^{-1}]) = (-1)^{\frac{k(k+1)}{2}} \sum_{l=0}^k (-1)^l \tau^{(0)}(t - 2[\lambda_l^{-1}]) \\
&\quad \times \text{Pf} \left(\left(\frac{\lambda_p - \lambda_q}{\lambda_p + \lambda_q} \right) \frac{\tau^{(0)}(t - 2[\lambda_p^{-1}] - 2[\lambda_q^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq p, q \leq k, p \neq l, q \neq l}. \quad (86)
\end{aligned}$$

We summarize the result above into the following proposition

Proposition 5. *The initial tau function $\tau^{(0)}(t)$ for the BKP hierarchy satisfies the higher Fay-like identities.*

- k is odd

$$\begin{aligned} & \prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \tau^{(0)}(t - 2 \sum_{j=0}^k [\lambda_j^{-1}]) \\ = & (-1)^{\frac{k(k+1)}{2}} \tau^{(0)}(t) \text{Pf} \left(\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \frac{\tau^{(0)}(t - 2[\lambda_i^{-1}] - 2[\lambda_j^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq i, j \leq k}. \end{aligned} \quad (87)$$

- k is even

$$\begin{aligned} & \prod_{i>j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \tau^{(0)}(t - 2 \sum_{j=0}^k [\lambda_j^{-1}]) = (-1)^{\frac{k(k+1)}{2}} \sum_{l=0}^k (-1)^l \tau^{(0)}(t - 2[\lambda_l^{-1}]) \\ & \times \text{Pf} \left(\left(\frac{\lambda_p - \lambda_q}{\lambda_p + \lambda_q} \right) \frac{\tau^{(0)}(t - 2[\lambda_p^{-1}] - 2[\lambda_q^{-1}])}{\tau^{(0)}(t)} \right)_{0 \leq p, q \leq k, p \neq l, q \neq l}. \end{aligned} \quad (88)$$

At last, let's conclude this section with some examples:

- For $k = 2$, (88) will lead to

$$\begin{aligned} & \frac{(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_0)}{(\lambda_2 + \lambda_0)(\lambda_2 + \lambda_1)(\lambda_1 + \lambda_0)} \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_1^{-1}] - 2[\lambda_2^{-1}]) \tau^{(0)}(t) \\ = & \frac{(\lambda_2 - \lambda_1)}{(\lambda_2 + \lambda_1)} \tau^{(0)}(t - 2[\lambda_1^{-1}] - 2[\lambda_2^{-1}]) \tau^{(0)}(t - 2[\lambda_0^{-1}]) \\ & - \frac{(\lambda_2 - \lambda_0)}{(\lambda_2 + \lambda_0)} \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_2^{-1}]) \tau^{(0)}(t - 2[\lambda_1^{-1}]) \\ & + \frac{(\lambda_1 - \lambda_0)}{(\lambda_1 + \lambda_0)} \tau^{(0)}(t - 2[\lambda_1^{-1}] - 2[\lambda_0^{-1}]) \tau^{(0)}(t - 2[\lambda_2^{-1}]), \end{aligned} \quad (89)$$

which is just (12) for $s_0 = -\lambda_0^{-1}, s_1 = -\lambda_1^{-1}, s_2 = -\lambda_2^{-1}, s_3 = 0$.

- For $k = 3$, according to (87),

$$\begin{aligned} & \frac{(\lambda_3 - \lambda_0)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_0)}{(\lambda_3 + \lambda_0)(\lambda_3 + \lambda_1)(\lambda_3 + \lambda_2)(\lambda_2 + \lambda_0)(\lambda_2 + \lambda_1)(\lambda_1 + \lambda_0)} \\ & \times \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_1^{-1}] - 2[\lambda_2^{-1}] - 2[\lambda_3^{-1}]) \tau^{(0)}(t) \\ = & \frac{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_3)}{(\lambda_0 + \lambda_1)(\lambda_2 + \lambda_3)} \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_1^{-1}]) \tau^{(0)}(t - 2[\lambda_2^{-1}] - 2[\lambda_3^{-1}]) \\ & - \frac{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_3)}{(\lambda_0 + \lambda_2)(\lambda_1 + \lambda_3)} \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_2^{-1}]) \tau^{(0)}(t - 2[\lambda_1^{-1}] - 2[\lambda_3^{-1}]) \\ & + \frac{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_2)}{(\lambda_0 + \lambda_3)(\lambda_1 + \lambda_2)} \tau^{(0)}(t - 2[\lambda_0^{-1}] - 2[\lambda_3^{-1}]) \tau^{(0)}(t - 2[\lambda_1^{-1}] - 2[\lambda_2^{-1}]), \end{aligned} \quad (91)$$

which is just (12) for $s_0 = -\lambda_0^{-1}, s_1 = -\lambda_1^{-1}, s_2 = -\lambda_2^{-1}, s_3 = -\lambda_3^{-1}$.

6. THE ADDITIONAL SYMMETRY AND THE GAUGE TRANSFORMATION FOR THE BKP HIERARCHY

In this section, we will study the compatibility between the additional symmetry and the gauge transformation of the BKP hierarchy.

Firstly, the additional symmetry of the BKP hierarchy (1) is defined by introducing the additional variables \hat{t}_{ml} [25]:

$$\hat{\partial}_{ml}L = [-(A_{ml}(M, L))_-, L], \quad \hat{\partial}_{ml}W = -(A_{ml}(M, L))_-W, \quad (92)$$

where

$$A_{ml}(M, L) = M^m L^l - (-1)^l L^{l-1} M^m L, \quad (93)$$

satisfying

$$A_{ml}(M, L)^* = -\partial A_{ml}(M, L)\partial^{-1} \quad (94)$$

and M is the Orlov-Schulman operator defined by

$$M = W\Gamma W^{-1}, \quad \Gamma = \sum_{n=0} (2n+1)t_{2n+1}\partial^{2n}, \quad (95)$$

which satisfies

$$\partial_{2n+1}M = [B_{2n+1}, M], \quad [L, M] = 1. \quad (96)$$

Before considering the additional symmetry under the gauge transformation, some lemmas are needed.

Lemma 6. *For any pseudo-differential operator P and a function f ,*

$$(f\partial f^{-1}Pf\partial^{-1}f^{-1})_+ = f\partial f^{-1}(P)_+f\partial^{-1}f^{-1} - (f\partial f^{-1}P_+)(f)\partial^{-1}f^{-1} \quad (97)$$

$$(f^{-1}\partial^{-1}fPf^{-1}\partial f)_- = f^{-1}\partial^{-1}f(P)_-f^{-1}\partial f - f^{-1}\partial^{-1}(f\partial f^{-1}P_+^*)(f) \quad (98)$$

Lemma 7. *For the eigenfunction Φ of the BKP hierarchy,*

$$A_{ml}(M, L)_+(\Phi) = -(T_D(\Phi)A_{ml}(M, L)T_D(\Phi)^{-1})_+^*(\Phi) \quad (99)$$

Proof. According to (94) and (97),

$$\begin{aligned} & (T_D(\Phi)A_{ml}T_D(\Phi)^{-1})_+^* \\ &= \left(T_D(\Phi)(A_{ml})_+T_D(\Phi)^{-1} - (T_D(\Phi)(A_{ml})_+)(\Phi)\partial^{-1}\Phi^{-1} \right)^* \\ &= T_I(\Phi)(A_{ml})_+^*T_I(\Phi)^{-1} + \Phi^{-1}\partial^{-1}(T_D(\Phi)(A_{ml})_+)(\Phi) \\ &= -T_I(\Phi)\partial(A_{ml})_+\partial^{-1}T_I(\Phi)^{-1} + \Phi^{-1}\partial^{-1}(T_D(\Phi)(A_{ml})_+)(\Phi), \end{aligned}$$

then

$$\left(-T_I(\Phi)\partial(A_{ml})_+\partial^{-1}T_I(\Phi)^{-1} \right)(\Phi) = -2\Phi^{-1} \int \Phi \partial_x \left((A_{ml})_+(\Phi) \right)$$

and

$$\begin{aligned} & \left(\Phi^{-1}\partial^{-1}(T_D(\Phi)(A_{ml})_+)(\Phi) \right)(\Phi) = \Phi^{-1} \int \Phi^2 \partial_x \left(\Phi^{-1}(A_{ml})_+(\Phi) \right) \\ &= -\Phi^{-1} \int (A_{ml})_+(\Phi) \Phi_x + \Phi^{-1} \int \Phi \partial_x \left((A_{ml})_+(\Phi) \right). \end{aligned}$$

Therefore

$$\begin{aligned}
& (T_D(\Phi)A_{ml}(M, L)T_D(\Phi)^{-1})_+^*(\Phi) \\
&= -\Phi^{-1} \int \left((A_{ml})_+(\Phi)\Phi_x + \Phi\partial_x((A_{ml})_+(\Phi)) \right) = -(A_{ml})_+(\Phi)
\end{aligned}$$

□

After the preparation above, we can get the proposition below.

Proposition 8. *Additional symmetry flows (92) commute with the gauge transformation $T = T(\Phi)$ for the BKP hierarchy, that is,*

$$\hat{\partial}_{ml}\tilde{L} = [-(A_{ml}(\tilde{M}, \tilde{L}))_-, \tilde{L}], \quad \partial_{ml}\tilde{W} = -(A_{ml}(\tilde{M}, \tilde{L}))_-\tilde{W}, \quad (100)$$

where

$$\tilde{L} = TLT^{-1}, \quad \tilde{M} = TMT^{-1}, \quad (101)$$

if and only if the eigenfunction Φ transforms under the additional symmetries as:

$$\hat{\partial}_{ml}\Phi = (A_{ml}(M, L))_+\Phi. \quad (102)$$

Proof. Firstly, by (92)

$$\hat{\partial}_{ml}\tilde{L} = \hat{\partial}_{ml}T \cdot LT^{-1} + T\hat{\partial}_{ml}L \cdot T^{-1} - TLT^{-1} \cdot \hat{\partial}_{ml}T \cdot T^{-1} \quad (103)$$

$$= [-T(A_{ml}(M, L))_-T^{-1} + \partial_{ml}T \cdot T^{-1}, \tilde{L}] \quad (104)$$

On the other hand,

$$[-(A_{ml}(\tilde{M}, \tilde{L}))_-, \tilde{L}] = [-(TA_{ml}(\tilde{M}, \tilde{L})T^{-1})_-, \tilde{L}]. \quad (105)$$

Thus (100) holds if and only if

$$-(TA_{ml}(M, L)T^{-1})_- = -T(A_{ml}(M, L))_-T^{-1} + \hat{\partial}_{ml}T \cdot T^{-1} \quad (106)$$

Since $T = T_I(\Phi/2)T_D(\Phi) = T_I(\Phi)T_D(\Phi)$, thus from (97), (98), and (99)

$$\begin{aligned}
& -(TA_{ml}(M, L)T^{-1})_- \\
&= -T_I(\Phi)(T_D(\Phi)A_{ml}(M, L)T_D(\Phi)^{-1})_-T_I(\Phi)^{-1} \\
& \quad + \Phi^{-1}\partial^{-1}\left(T_D(\Phi)(T_D(\Phi)A_{ml}(M, L)T_D(\Phi)^{-1})_+^*\right)(\Phi) \\
&= -T_I(\Phi)(T_D(\Phi)A_{ml}(M, L)T_D(\Phi)^{-1})_-T_I(\Phi)^{-1} - \Phi^{-1}\partial^{-1}\left(T_D(\Phi)(A_{ml}(M, L))_+\right)(\Phi) \\
&= -T(A_{ml}(M, L))_-T^{-1} - T_I(\Phi)\left(T_D(\Phi)(A_{ml}(M, L))_+\right)(\Phi)\partial^{-1}\Phi^{-1}T_I(\Phi)^{-1} \\
& \quad - \Phi^{-1}\partial^{-1}\left(T_D(\Phi)(A_{ml}(M, L))_+\right)(\Phi).
\end{aligned} \quad (107)$$

On the other hand,

$$\begin{aligned}
\hat{\partial}_{ml}T \cdot T^{-1} &= \hat{\partial}_{ml}T_I(\Phi) \cdot T_I(\Phi)^{-1} + T_I(\Phi)\hat{\partial}_{ml}T_D(\Phi) \cdot T_D(\Phi)^{-1}T_I(\Phi)^{-1} \\
&= -\Phi^{-1}\hat{\partial}_{ml}\Phi + \Phi^{-1}\partial^{-1}\left(\hat{\partial}_{ml}\Phi\right)\Phi^{-1}\partial\Phi + T_I(\Phi) \cdot \left(\hat{\partial}_{ml}\Phi\right)\Phi^{-1} \cdot T_I(\Phi)^{-1} \\
&\quad -T_I(\Phi) \cdot \Phi\partial\Phi^{-2} \cdot \left(\hat{\partial}_{ml}\Phi\right)\Phi\partial^{-1}\Phi^{-1} \cdot T_I(\Phi)^{-1} \\
&= -\Phi^{-1}\partial^{-1}\left(T_D(\Phi)(\hat{\partial}_{ml}\Phi)\right) - T_I(\Phi) \cdot \left(T_D(\Phi)(\hat{\partial}_{ml}\Phi)\right)\partial^{-1}\Phi^{-1} \cdot T_I(\Phi)^{-1} \quad (108)
\end{aligned}$$

Therefore from (107) and (108), (106) is true if and only if

$$\hat{\partial}_{ml}\Phi = (A_{ml}(M, L))_+\Phi.$$

□

7. CONCLUSIONS AND DISCUSSIONS

We have provided four applications of the gauge transformation for the BKP hierarchy.

- The orbit of the gauge transformation for the constrained BKP hierarchy defines a special $(2+1)$ -dimensional Toda lattice equation structure (see Proposition 2);
- Starting from the Grammian determinant solutions of the BKP hierarchy, we derived two types of the Pfaffian structures for the BKP hierarchy (see Proposition 4);
- Through the gauge transformation of the BKP hierarchy, the higher order Fay-like identities are derived (see Proposition 5);
- The compatibility between the additional symmetry and the gauge transformation of the BKP hierarchy is studied (see Proposition 8).

Our results show that the gauge transformation is also convenient tool to study integrable properties, such as the Fay-like identities and the additional symmetries for the BKP hierarchy.

For the non-Hermitian random matrix models, there are the Pfaffian tau function structures [38, 39], which are closely related with the BKP hierarchy. We hope that our results will be helpful for the comprehension of the link between the BKP hierarchy and the non-Hermitian random matrix models.

Acknowledgements: This work is supported by the NSFC (Grant No. 11226196) and “the Fundamental Research Funds for the Central Universities” No. 2012QNA45.

REFERENCES

- [1] L. L. Chau, J. C. Shaw and H. C. Yen. Solving the KP hierarchy by gauge transformations. Comm. Math. Phys. 149(1992) 263-278.
- [2] W. Oevel. Darboux theorems and Wronskian formulas for integrable system I: constrained KP flows. Physica A 195 (1993) 533-576.
- [3] H. Aratyn, E. Nissimov and S. Pacheva. Darboux-Backlund solutions of $SL(p, q)$ KP-KdV hierarchies, constrained generalized Toda lattices, and two-matrix string model. Phys. Lett. A. 201(1995) 293-305 (arXiv:hep-th/9501018).

- [4] L. L. Chau, J. C. Shaw and M. H. Tu. Solving the constrained KP hierarchy by gauge transformations. J. Math. Phys. 38 (1997) 4128-4137.
- [5] R. Willox, I. Loris, C. R. Gilson. Binary Darboux transformations for constrained KP hierarchies. Inverse Problems 13 (1997) 849-865.
- [6] J. S. He, Y. S. Li and Y. Cheng. Two choices of the gauge transformation for the AKNS hierarchy through the constrained KP hierarchy. J. Math. Phys. 44 (2003) 3928 -3960.
- [7] J. J. C. Nimmo. Darboux transformations from reductions of the KP hierarchy, in *Nonlinear Evolution Equations and Dynamical Systems*, edited by V. G. Makhankov, A. R. Bishop, and D. D. Holm, World Scientific, Singapore, 1995, 168-177. arXiv:solv-int/9410001.
- [8] J. S. He, Y. Cheng and R. A. Roemer. Solving bi-directional soliton equations in the KP hierarchy by gauge transformation. J. High Energy Phys. 03(2006) 103. arXiv:math-ph/0503028.
- [9] J. S. He, Z. W. Wu and Y. Cheng. Gauge transformations for the constrained CKP and BKP hierarchies. J. Math. Phys. 48 (2007) 113519.
- [10] J. S. He and X. D. Li. Solutions of the $(2 + 1)$ -dimensional KP, SK and KK equations generalized by gauge transformations from nonzero seeds. J. Nonlin. Math. Phys. 2 (2009) 179-194.
- [11] W. Oevel. Darboux transformations for integrable lattice systems, in *Nonlinear Physics: theory and experiment*, ed. by E. Alfinito, L. Martina and F. Pempinelli, World Scientific, Singapore, 1996, 233-240.
- [12] S. W. Liu, Y. Cheng and J. S. He. The determinant representation of the gauge transformation for the discrete KP hierarchy. Science China Mathematics 53 (2010) 1195-1206 (arXiv:0904.1868).
- [13] M. H. Tu, J. C. Shaw, and C. R. Lee. On Darboux - Backlund transformations for the q -deformed Korteweg - de Vries hierarchy. Lett. Math. Phys. 49 (1999) 33 - 45 (arXiv:solv-int/9811004).
- [14] J. S. He, Y. H. Li and Y. Cheng. q -deformed KP hierarchy and q -deformed constrained KP hierarchy. SIGMA 2 (2006) 060.
- [15] E. Date, M. Jimbo, M. Kashiwara and T. Miwa. Transformation groups for soliton equations, in *Nonlinear integrable systems - classical theory and quantum theory*, ed. by M. Jimbo and T. Miwa, World Scientific, Singapore, 1983, pp.39-119.
- [16] C. W. Cao, X. G. Geng and Y. T. Wu. From the special $2 + 1$ Toda lattice to the Kadomtsev-Petviashvili equation. J. Phys. A: Math. Gen. 32(1999) 8059-8078.
- [17] X. M. Qian, S. Y. Lou and X. B. Hu. Variable separation approach for a differential-difference system: special Toda equation. J. Phys. A: Math. Gen. 37 (2004) 2401C2411.
- [18] H. W. Tam, X. B. Hu and X. M. Qian. Remarks on several 2?1 dimensional lattices. J. Math. Phys. 43 (2002) 1008-1017.
- [19] J. B. Chen and Z. J. Qiao. Darboux transformation of the special $(2 + 1)$ -dimensional Toda lattice and its explicit solution. Phys. Scr. 82 (2010) 015003.
- [20] R. Hirota. Soliton solutions to the BKP equations. I. the Pfaffian technique. J. Phys. Soc. Jpn. 58 (1989) 2285-2296.
- [21] I. Loris and R. Willox. Symmetry reductions of the BKP hierarchy. J. Math. Phys. 40 (1999) 1420-1431.
- [22] I. Loris and R. Willox. On solutions of constrained Kadomtsev- Petviashvili equations: Grammians. J. Math. Phys. 38 (1997) 5190-5197.
- [23] P. van Moerbeke. Integrable fundations of string theory, in *Lectures on Integrable systems*, Edited by O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach, World Scientific, Singapore, 1994, pp.163-267.
- [24] J. van de Luer. Matrix integrals and the geometry of spinors. J. Nonlin. Math. Phys. 8 (2001) 288-311.

- [25] M. H. Tu. On the BKP hierarchy: Additional symmetries, Fay identity and Adler-Shiota- van Moerbeke formula. *Lett. Math. Phys.* 81 (2007) 91-105.
- [26] J. P. Cheng, J. S. He and S. Hu. The “ghost” symmetry of the BKP hierarchy. *J. Math. Phys.* 51(2010) 053514.
- [27] A. Yu. Orlov and E. I. Schulman. Additional symmetries for integrable systems and conformal algebra representation. *Lett. Math. Phys.* 12 (1993) 171-179.
- [28] K. Takasaki. Quasi-Classical Limit of BKP Hierarchy and W-Infinity Symmetries, *Lett.Math.Phys.* 28(1993), 117-185.
- [29] M. Adler, T. Shiota and P. van Moerbeke. A Lax representation for the vertex operator and the central extension. *Comm. Math. Phys.* 171 (1995) 547-588.
- [30] L. A. Dickey. On additional symmetries of the KP hierarchy and Sato’s Bäcklund transformation. *Comm. Math. Phys.* 167 (1995) 227-233.
- [31] J. S. He, K. L. Tian, A. Foerster and W. X. Ma. Additional symmetries and string equation of the CKP hierarchy. *Lett. Math. Phys.* 81 (2007) 119-134.
- [32] J. P. Cheng, K. L. Tian and J. S. He. The additional symmetries for the BTL and CTL hierarchies. *J. Math. Phys.* 52(2011) 053515.
- [33] C. Z. Li and J. S. He. Dispersionless bigraded Toda hierarchy and its additional symmetry. *Rev. Math. Phys.* 24 (2012) 1230003.
- [34] K. L. Tian, J. S. He, J. P. Cheng and Y. Cheng. Additional symmetries of constrained CKP and BKP hierarchies. *Sci. China Math.* 54 (2011) 257-268.
- [35] Y. Cheng. Constraints of the Kadomtsev-Petviashvili hierarchy. *J. Math. Phys.* 33 (1992) 3774C3782.
- [36] Q. W. Wang. The core ideas and applications of the linear algebra (in Chinese). Science Press, Beijing, 2012.
- [37] R. Hirota. The Direct Method in Soliton Theory. Cambridge University Press, 2004.
- [38] M. Adler, T. Shiota and P. van Moerbeke. Pfaff τ - functions. arXiv: solv-int/9909010.
- [39] M. Adler and P. van Moerbeke. Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum. *Ann. Math.* 153 (2001) 149-189.